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THE RADON–NIKODÝM PROPERTY AND WEIGHTED TREES IN BANACH SPACES†

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ABSTRACT

It is known that a *dual* Banach space has the Radon–Nikodým Property if and only if it does not have a tree. It is shown that a Banach space has the Radon–Nikodým Property if and only if it does not have a "weighted tree".

It has long been known that l_1 does not contain a tree [2, pp. 163–165]. In fact, a necessary and sufficient condition for a dual Banach space to have the Radon–Nikodým Property (RNP) is that it contains no tree. The necessity follows from nondentability of the closed convex span of a tree, and is valid for all Banach spaces. For the sufficiency, Stegall has proved that every nondentable *dual* space contains a tree [3, pp. 218–219]. It is well known that a general Banach space has RNP if and only if it is dentable. For a summary of known results on RNP, see [1].

It is an open question whether a non-dual space without RNP must have a tree. We will show that the absence of a weighted tree is a necessary and sufficient condition for a Banach space to be dentable, or to have RNP.

A *dentable* subset of a Banach space is a subset K such that, for every $\varepsilon > 0$, there is a point p in K such that p is not contained in $\overline{\text{co}}[K \setminus B(p, \varepsilon)]$, where $\overline{\text{co}}[M]$ is the closed convex span of M and $B(p, \varepsilon)$ is the ε-ball centered at p. A Banach space is said to be *dentable* if every bounded subset is dentable. It follows that a Banach space is dentable if and only if every bounded closed convex subset is dentable.

DEFINITION. A *tree* in a Banach space is a subset

$$T = \{x^{ni} : 1 \leqq i \leqq 2^{n-1};\ n = 1, 2, \cdots\}$$

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satisfying the conditions:

(T1) $\|x^{ni}\| \leqq 1$.

(T2) $x^{ni} = \frac{1}{2}(x^{n+1,2i-1} + x^{n+1,2i})$ for all n and i.

(T3) There is a *separation constant* $\varepsilon > 0$, which means that, for all n and i,

$$\|x^{n+1,2i-1} - x^{n+1,2i}\| > \varepsilon.$$

It is easy to see that a Banach space is not dentable if it contains a tree, since the sets T and $\overline{\mathrm{co}}[T]$ are not dentable.

DEFINITION. A *branch* of a tree is a sequence $\{w_n\}$ such that each w_n is x^{ni} for some value of i, and if $w_n = x^{ni}$, then $w_{n+1} = x^{n+1,2i-1}$ or $x^{n+1,2i}$. An *eventually left-turning branch* of a tree is a branch $\{w_n\}$ for which there is a positive integer k such that, for each $n \geqq k$, if $w_n = x^{ni}$ on the branch, then

$$w_{n+1} = x^{n+1,2i-1}.$$

DEFINITION. A *weighted tree* in a Banach space is a subset

$$T = \{x^{ni} : 1 \leqq i \leqq 2^{n-1};\ n = 1, 2, \cdots\}$$

satisfying the following conditions, where

$$\{t_{ni} : 1 \leqq i \leqq 2^{n-1};\ n = 1, 2, \cdots\}$$

is a set of real numbers for which $0 \leqq t_{ni} \leqq \frac{1}{2}$ and $\Sigma_{n=1}^{\infty} t_n = \infty$ if $\{t_n\}$ is a subset of $\{t_{ni}\}$ corresponding to points along an eventually left-turning branch.

(T1) $\|x^{ni}\| \leqq 1$.

(WT2) $x^{ni} = (1 - t_{ni})x^{n+1,2i-1} + t_{ni}x^{n+1,2i}$ for all n and i.

(T3) There is a *separation constant* $\varepsilon > 0$, which means that, for all n and i,

$$\|x^{n+1,2i-1} - x^{n+1,2i}\| > \varepsilon.$$

DEFINITION. An *approximate tree* in a Banach space is a subset

$$T = \{x^{ni} : 1 \leqq i \leqq 2^{n-1};\ n = 1, 2, \cdots\}$$

satisfying the conditions:

(T1) $\|x^{ni}\| \leqq 1$.

(AT2) There is a sequence of non-negative numbers $\{\delta_n\}$, the *errors in averaging*, such that $\Sigma_{n=1}^{\infty} \delta_n < \infty$ and, for each n,

$$\|x^{ni} - \tfrac{1}{2}(x^{n+1,2i-1} + x^{n+1,2i})\| < \delta_n.$$

(T3) There is a *separation constant* $\varepsilon > 0$ such that, for all n and i,

$$\|x^{n+1,2i-1} - x^{n+1,2i}\| > \varepsilon.$$

A "tree" is a "weighted tree" for which each $t_{ni} = \frac{1}{2}$, and also an "approximate tree" whose "errors in averaging" are all zero.

The definition of weighted tree can be modified in a way analogous to the definition of approximate tree, to obtain a definition of *approximate weighted tree.*

LEMMA 1. *If a Banach space X has an approximate (weighted) tree, then X has a (weighted) tree.*

PROOF. Let $T = \{x^{ni}\}$ be an approximate tree with separation constant $\varepsilon > 0$ and errors in averaging $\{\delta_n\}$. Since one can use a subtree of the given tree, there is no loss of generality if we assume that $\sum_{m=1}^{\infty} \delta_m < \frac{1}{2}\varepsilon$. Let

$$x_k^{ni} = 2^{-k} \sum \{x^{n+k,j}: 2^k \cdot (i-1) + 1 \leqq j \leqq 2^k \cdot i\}.$$

Then

$$\|x_p^{ni} - x_q^{ni}\| < \sum_{m=p}^{q-1} \delta_{m+n}.$$

Thus $\{x_k^{ni}: k \geqq 1\}$ converges to some element y^{ni} as $k \to \infty$. We have

$$x_k^{ni} = \tfrac{1}{2}(x_{k-1}^{n+1,2i-1} + x_{k-1}^{n+1,2i})$$

and, for all k,

$$\|y^{ni} - x_k^{ni}\| < \delta = \sum_{m=1}^{\infty} \delta_m.$$

Thus $\{y^{ni}\}$ is a tree with separation constant at least $\varepsilon - 2\delta$, which is positive.

By using weights and functions of weights instead of $\frac{1}{2}$ and 2^{-k}, this proof can be adapted to approximate weighted trees.

We proceed to prove that a general Banach space X is nondentable if and only if X contains a weighted tree. First, we establish three lemmas.

LEMMA 2. *If a Banach space X contains a weighted tree, then X is not dentable.*

PROOF. Let $T = \{x^{ni}\}$ be a weighted tree with separation constant $\varepsilon > 0$ and weights $\{t_{ni}\}$ such that

$$0 \leqq t_{ni} \leqq \tfrac{1}{2} \quad \text{and} \quad \sum t_{ni} = \infty$$

if the sum is taken along any branch that is eventually left-turning. Suppose the Banach space X is dentable. Then, for a positive number $\frac{1}{4}\varepsilon$, there is an x^{ni} in T, which we shall call x^1, such that

$$x^1 \notin K = \overline{\mathrm{co}}[T \setminus B(x^1, \tfrac{1}{4}\varepsilon)]. \tag{1}$$

Since K is bounded, closed, and convex, there is a positive number η such that $\mathrm{dist}[B(x^1,\eta),K]>0$. Therefore, there is a continuous linear functional f, a number θ, and a positive number α, such that

$$f(y)<\theta \quad \text{if} \quad y \in K, \qquad f(x^1)>\alpha+\theta.$$

It follows from (1) that the diameter of H is at most $\frac{1}{2}\varepsilon$, where

$$H=\{x \in T : f(x) \geqq \theta\}.$$

We have $x^1=(1-t_{ni})x^{n+1,2i-1}+t_{ni}x^{n+1,2i}$. Thus

$$x^1-x^{n+1,2i}=(1-t_{ni})(x^{n+1,2i-1}-x^{n+1,2i}),$$

and

$$\|x^1-x^{n+1,2i}\|>\varepsilon(1-t_{ni}) \geqq \tfrac{1}{2}\varepsilon. \tag{2}$$

Since $x^1 \in H$ and the diameter of H is at most $\frac{1}{2}\varepsilon$, we have $x^{n+1,2i} \notin H$. Therefore $f(x^{n+1,2i})<\theta$. Let $x^2=x^{n+1,2i-1}$. Since

$$\begin{aligned}\alpha+\theta<f(x^1)&=(1-t_{ni})f(x^2)+t_{ni}f(x^{n+1,2i})\\&<(1-t_{ni})f(x^2)+\theta t_{ni},\end{aligned}$$

division by $1-t_{ni}$ gives $f(x^2)>\alpha(1-t_{ni})^{-1}+\theta$. Thus $x^2 \in H$.

Inductively, we obtain $\{x^k\}$ such that $x^k \in H$, and

$$f(x^{k+1})>\alpha \prod_{m=1}^{k}(1-\omega_m)^{-1}+\theta,$$

where $\omega_1=t_{ni}$ and $\{\omega_m\}$ is a sequence of successive values of $t_{n+m-1,j}$ along an eventually left-turning branch through x^{ni}.

Since $\Sigma_{m=1}^{\infty}\omega_m=\infty$, $\Pi_{m=1}^{\infty}(1-\omega_m)=0$. Thus f is not bounded on the unit ball of X. Hence T is not dentable, which implies that the space X is not dentable.

LEMMA 3. *Let ε and η be positive numbers such that*

$$\eta<\frac{1}{2}, \quad \varepsilon>2 \quad \textit{and} \quad \frac{\eta}{1-2\eta}<\frac{\varepsilon}{8}.$$

Then $\eta<\varepsilon/8<\frac{1}{4}$. Let x be a point in the closed unit ball of a Banach space X and suppose that, for some $n \geqq 2$, there is a subset $\{x_i : 1 \leqq i \leqq n\}$ of the unit ball of X and a monotone decreasing sequence of positive numbers $\{t_i : 1 \leqq i \leqq n\}$ such that $x=\Sigma_{i=1}^{n} t_i x_i$ and

(B1) $\|x_i\| \leqq 1$ *for all* i,

(B2) $\sum_{i=1}^{n} t_i = 1$,

(B3) $\|x - x_i\| > \varepsilon$ *for all* i.

Then there is a positive integer k, *a finite sequence of positive numbers* $\{s_m : 1 \leqq m \leqq k\}$, *and finite sequences* $\{y_1, y_2, \cdots, y_k\}$ *and* $\{z_1, z_2, \cdots, z_k\}$ *in* $\operatorname{co}\{x_1, \cdots, x_n\}$, *which satisfy the following conditions for all* m *such that* $1 \leqq m \leqq k$:

(i) $\|y_m\| \leqq 1$ *and* $\|z_m\| \leqq 1$,

(ii) $\|z_m - y_m\| > \frac{1}{2}\varepsilon$,

(iii) $y_{m-1} = (1 - s_m)y_m + s_m z_m$, *where* $y_0 = x$,

(iv) $\frac{1}{2} \geqq \sum_{m=1}^{k} s_m > \eta$.

PROOF. *Case* 1: $t_1 > \eta$. In this case, we shall see that k can be 1. If we let

$$w = \frac{\sum_{i=2}^{n} t_i x_i}{\sum_{i=2}^{n} t_i} = \frac{\sum_{i=2}^{n} t_i x_i}{1 - t_1},$$

then $x = t_1 x_1 + (1 - t_1)w$ and we have

$$\|x_1 - w\| = \frac{\|x - x_1\|}{1 - t_1} > \varepsilon,$$

so that (ii) is satisfied if (y_1, z_1) is either (x_1, w) or (w, x_1). We also have

$$\begin{aligned} \varepsilon < \|x - x_1\| &\leqq (1 - t_1) + \|x - t_1 x_1\| \\ &= (1 - t_1) + \left\| \sum_{2}^{n} t_i x_i \right\| \leqq 2(1 - t_1). \end{aligned}$$

Thus $1 - t_1 > \frac{1}{2}\varepsilon > \eta$. Now let $s_1 = \min\{1 - t_1, t_1\}$. Then $\eta < s_1 \leqq \frac{1}{2}$. If $s_1 = t_1$, let $y_1 = w$ and $z_1 = x_1$. On the other hand, if $s_1 = 1 - t_1$, let $y_1 = x_1$ and $z_1 = w$.

Case 2: $t_1 \leqq \eta$. Since $\sum_{i=1}^{n} t_i = 1$, there is a positive integer $k < n$ such that

$$\eta < t_1 + t_2 + \cdots + t_k \leqq 2\eta.$$

For each m such that $1 \leqq m \leqq k$, let

$$s_m = \frac{t_m}{\sum_{i=m}^{n} t_i}.$$

Clearly,

$$\sum_{m=1}^{k} s_m > \sum_{m=1}^{k} t_m > \eta. \tag{3}$$

Also,

$$s_m < \frac{t_m}{\sum_{i=k+1}^{n} t_i} = \frac{t_m}{1 - \sum_{i=1}^{k} t_i} \leqq \frac{t_m}{1-2\eta}. \tag{4}$$

Thus

$$\sum_{m=1}^{k} s_m < \sum_{m=1}^{k} \frac{t_m}{1-2\eta} \leqq \frac{2\eta}{1-2\eta} < \frac{1}{4}\varepsilon < \frac{1}{2}.$$

This and (3) imply that (iv) is satisfied. Let

$$y_{m-1} = \frac{\sum_{i=m}^{n} t_i x_i}{\sum_{i=m}^{n} t_i}. \tag{5}$$

Then $y_0 = x$. Let $z_m = x_m$. Therefore

$$y_{m-1} = (1 - s_m)y_m + s_m x_m \tag{6}$$

implies (iii) is satisfied. Clearly (i) is satisfied. It remains to show that $\| z_m - y_m \| = \| x_m - y_m \| > \frac{1}{2}\varepsilon$. We shall show first that $\| x - y_m \| < \frac{1}{2}\varepsilon$. This follows from the definition of s_m, (4), and (5), since

$$\begin{aligned} \| x - y_m \| &\leqq \| x - y_1 \| + \sum_{i=2}^{m} \| y_{i-1} - y_i \| \\ &= \sum_{i=1}^{m} s_i \| x_i - y_i \| \leqq 2 \sum_{i=1}^{m} s_i \\ &\leqq 2 \sum_{i=1}^{m} \frac{t_i}{1-2\eta} \leqq \frac{4\eta}{1-2\eta} \\ &< \tfrac{1}{2}\varepsilon. \end{aligned}$$

We then have

$$\| y_m - x_m \| \geqq \| x - x_m \| - \| x - y_m \| > \varepsilon - \tfrac{1}{2}\varepsilon = \tfrac{1}{2}\varepsilon.$$

LEMMA 4. *If a Banach space X is not dentable, then X contains a weighted tree.*

PROOF. If X is not dentable, then X contains a bounded closed convex nonvoid subset K for which there is a positive number ε such that, for each $p \in K$, $p \in \overline{\text{co}}[K \setminus B(p, 2\varepsilon)]$, i.e., K is not 2ε-dentable. Without loss of generality, we can assume K is a subset of the unit ball of X. This implies $\varepsilon < 1$. Let η be a positive number for which

$$\eta < \frac{1}{2} \quad \text{and} \quad \frac{\eta}{1-2\eta} < \frac{\varepsilon}{8}.$$

Let $\{\delta_n\}$ be a sequence of positive numbers such that $\delta_n < \varepsilon$ for each n, and

$$\sum_{n=1}^{\infty} \delta_n < \infty.$$

Let p be an arbitrary point of K. Since K is not 2ε-dentable, for this ε, this η, and this δ_1, there is a subset $\{x_1, x_2, \cdots, x_n\}$ of K such that

$$\left\| p - \sum_{i=1}^{n} t_i x_i \right\| < \delta_1 < \varepsilon,$$

and $\|p - x_i\| > 2\varepsilon$ for each i, where $\{t_i\}$ is a monotone decreasing sequence of positive numbers with $\Sigma_{i=1}^{n} t_i = 1$. Let $x = \Sigma_{i=1}^{n} t_i x_i$. Then conditions (B1) through (B3) in Lemma 3 are satisfied, since

$$\|x - x_i\| \geqq \|x_i - p\| - \|x - p\| > \varepsilon.$$

Therefore there is a positive integer k, and finite sequences $\{y_1, y_2, \cdots, y_k\}$ and $\{z_1, z_2, \cdots, z_k\}$ contained in the convex span of $\{x_1, \cdots, x_n\}$, such that properties (i) through (iv) of Lemma 3 are satisfied. Now, for $1 \leqq m \leqq k$, let $x^{1,1} = p$, $x^{m+1,1} = y_m$, $x^{m+1,2} = z_m$, and $t_{m,1} = s_m$. Iterate this process with p successively being $z_1, z_2, \cdots, z_k$, and finally y_k, using $\delta_2, \delta_3, \cdots, \delta_{k+1}, \delta_{k+1}$, respectively, instead of δ_1. The natural inductive continuation of this process yields an approximate weighted tree with separation constant $\frac{1}{2}\varepsilon$ and errors of averaging $\{\delta_n\}$. It follows from Lemma 1 that X has a weighted tree.

THEOREM A. *A Banach space X contains a weighted tree if and only if X is not dentable.*

Theorem A follows directly from Lemma 2 and Lemma 4. Moreover, since RNP is equivalent to dentability for Banach spaces [1, theorem, p. 25], we have the following:

THEOREM B. *A Banach space has RNP if and only if it contains no weighted tree.*

There are other definitions of "weighted tree" that are stronger than our definition, but formally weaker than the concept of a tree. For example, one might assume that $t_{ni} = t_n$, independent of i, and that $\Sigma_{n=1}^{\infty} t_n = \infty$. Existence of such "weighted trees" implies the space does not have RNP, but it is not known whether such existence is implied by the space not having RNP.

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